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On Bifurcations Inducing a Strange Attractor in the System of A. M. Oboukhov

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The paper contains a numerical study of qualitative properties of motion in a dynamical system modeling a turbulent flow. It shows that after four bifurcations related to a growing active force there appears a strange attractor in the system and the motion becomes stochastic.

KEY WORDS: Turbulent flow; stochastic motion; strange attractor.

Oboukhov[(1)] introduced a quadratic nonlinear system to model a cascade mechanism of energy transformation in a turbulent flow,

$$\begin{aligned} \dot{u}_0 &= -u_{1,1}^2 - u_0 + F \\ \dot{u}_{1,1} &= u_0 u_{1,1} + Q(u_{2,1}^2 - u_{2,2}^2) - Q^2 u_{1,1} \\ &\vdots \\ \dot{u}_{i,2k-1} &= -Q^{i-1} u_{i-1,k} u_{i,2k-1} + Q^i(u_{i+1,4k-3}^2 - u_{i+1,4k-2}^2) - Q^{2i} u_{i,2k-1} \\ \dot{u}_{i,2k} &= Q^{i-1} u_{i-1,k} u_{i,2k} + Q^i(u_{i+1,4k-1}^2 - u_{i+1,4k}^2) - Q^{2i} u_{i,2k} \end{aligned} \tag{1a}$$

for $i = 2, 3, \ldots, N-1$ and $1 \leqslant k \leqslant 2^{i-2}$; and

$$\begin{aligned} \dot{u}_{i,2k-1} &= -Q^{i-1} u_{i-1,k} u_{i,2k-1} - Q^{2i} u_{i,2k-1} \\ \dot{u}_{i,2k} &= Q^{i-1} u_{i-1,k} u_{i,2k} - Q^{2i} u_{i,2k} \end{aligned} \tag{1b}$$

for $i = N$ and $1 \leqslant k \leqslant 2^{N-2}$; where $Q > 1$ is the scale reduction parameter for the transition from the ith to the $(i+1)$th row; $F > 0$ is the force acting upon the system's largest scale mode.

When the acting force is very small ($F < Q^2$) the motion is of a regular character (a laminar flow). In the system's phase space this corresponds to the

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only steady-state point being $u_0{}^s = F$, $u_{i,k}^s = 0$ for every $i \geqslant 1$; k. A greater force causes a number of bifurcations to excite modes of an increasingly lower scale; the motion becomes more complicated and acquires a stochastic character (a turbulent flow). The first three bifurcations have been found by Glukhovsky[(2)]; they consist in the following.

With $F \geqslant Q^2$ the fixed point of the laminar flow loses its stability and, depending on the sign of the initial perturbation, there appears one of the two secondary stationary regimes [$u_0{}^s = Q^2$, $u_{1,1}^s = \pm(F - Q^2)^{1/2}$, $u_{i,k}^s = 0$ for every $i \geqslant 2$; k]. In turn, when $F \geqslant Q^6 + Q^2$, second-row modes get excited and one of the four stationary regimes is set up [$u_0{}^s = Q^2$, $u_{1,1}^s = \pm Q^3$, $u_{2,j}^s = \pm Q(F - Q^6 - Q^2)^{1/2}$, $u_{2,k}^s = 0, j = 1$ or $2, j \neq k$, $u_{i,k}^s = 0$ for every $i \geqslant 3$; k]. The third bifurcation, after which the system (1) has no steady-state solutions, can be observed with $F = Q^6 + Q^2 + q(Q)$, where $q(Q) < 1$, so that $(d/dQ)q(Q) > 0$, $\lim_{Q\to\infty} q(Q) = 1$.

Further analysis is to be accomplished numerically through the study of mappings of some hypersurfaces (or Poincaré maps) which are secants of the phase trajectories of system (1). Bearing in mind that (1) abides by a symmetry group in relation to all coordinates axes except for the senior one, it is sufficient for qualitative analysis of the situation after the third bifurcation to consider the phase space area where only three modes u_0, $u_{1,1}$, and $u_{2,1}$ are excited. Let the surface $u_0 = Q^{-2}u_{2,1}^2 + Q^2$ serve as a Poincaré map, since it is a secant for all trajectories in the area except for the separatrix going from the secondary flow steady point to the steady point emerging after the second bifurcation (this separatrix lies wholly in the Poincaré map).

Numerical integration of system (1) in these circumstances shows that after the above-mentioned third bifurcation there emerges one stable limitary cycle in the vicinity of every steady point that has lost its stability (a stable steady point serves as cycle image in the Poincaré map). If the active force increases further, these cycles lose their stability, and at the moment of the fourth bifurcation there appears around every periodic motion an infinite set of invariant toruses enclosed in one another, on which the system trajectories accomplish quasiperiodic motion. Simultaneously, a heteroclinic structure emerges which includes all steady saddle points and separatrices joining them (with the symmetry taken into account).

After the force exceeds the bifurcation value the invariant toruses disappear and an unstable limitary cycle replaces every stable one. The accompanying complicated motion is to be analyzed through the four simultaneously excited modes u_0, $u_{1,1}$, $u_{2,1}$, and $u_{2,2}$. Generally speaking, the Poincaré map for this case should be three-dimensional. But, as seen in (1), in case of the modes at rest of the third and the subsequent rows, when mode $u_{2,1}$ is excited, mode $u_{2,2}$ is being damped, and contrariwise. Hence, all mappings on the three-dimensional Poincaré map determined by the equation

$u_0 = Q^{-2}(u_{2,1}^2 + u_{2,2}^2) + Q^2$ in hyperspace will tend to either its two-dimensional sheet $u_0 = Q^{-2}u_{2,1}^2 + Q^2$ or to the sheet orthogonal to it, $u_0 = Q^{-2}u_{2,2}^2 + Q^2$. With this "separatrix" approximation of point mappings in the Poincaré map the system's motion may be represented as shown in Fig. 1.

The typical trajectory of a point mapping appears in the shape of a curve making a number of turns around an unstable focus, for instance, in area $u_{1,1} < 0, u_{2,1} > 0$, in the sheet $u_0 = Q^{-2}u_{2,1}^2 + Q^2$. The closer to the focus is the initial point, the greater is the number of turns. Then the trajectory is transferred into area $u_{1,1} > 0$, where mode $u_{2,1}$ is soon damped, while $u_{2,2}$, which was formerly close to a state of rest, is rapidly increasing in modulus (its sign is determined by the sign of the initial state at the moment of transition). Thereafter one has to consider the point mapping in the sheet $u_0 = Q^{-2}u_{2,2}^2 + Q^2$, where a similar picture is to be found, and so on.

The set of such trajectories is attracting, according to the separatrix approximation mentioned above. At the same time each individual trajectory is unstable. Its transition from the vicinity of one focus to the vicinity of another depends on subtle details of prehistory and is therefore unpredictable if the accuracy of the modes' current coordinates is finite. As a result, as calculations show, the disjunction of temporal correlations in the system in question takes up the time of a trajectory return to the Poincaré map. It is also noteworthy that the graph of subsequent changes in the mode u_0 fluctuation amplitude reveals that along with the typical nonperiodic trajectory described earlier the attracting set contains dense periodic trajectories everywhere. Thus, the set in question cannot be differentiated in principle from Lorenz's strange attractor.[3,4]

We note in conclusion that with a further increase of the acting force

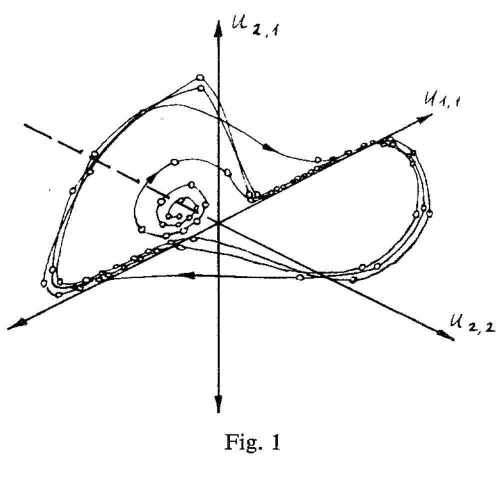

Fig. 1

(under $F > 2Q^6 + Q^2$) modes of the third and the subsequent rows are excited. Judging by the character of temporal correlations, and by the positiveness of topological entropy determined through the symmetricized Jacobi matrix maximum eigenvalue of the right parts of system (1), as well as by the unpredictability of future phase coordinates with approximately given initial data, the basic peculiarities of the stochastic behavior of system (1) found in a strange attractor are preserved.

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